

The Shatashvili-Vafa G_2 superconformal algebra as a Quantum Hamiltonian Reduction of $D(2, 1; \alpha)$.

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Abstract

We obtain the superconformal algebra associated to a sigma model with target a manifold with G_2 holonomy, i.e., the Shatashvili-Vafa G_2 algebra as a quantum Hamiltonian reduction of the exceptional Lie superalgebra $D(2, 1; \alpha)$ for $\alpha = 1$. We produce the complete family of W -algebras $SW(\frac{3}{2}, \frac{3}{2}, 2)$ (extensions of the $N = 1$ superconformal algebra by two primary supercurrents of conformal weight $\frac{3}{2}$ and 2 respectively) as a quantum Hamiltonian reduction of $D(2, 1; \alpha)$. As a corollary we find a free field realization of the Shatashvili-Vafa G_2 algebra, and an explicit description of the screening operators.

1 Introduction

The Shatashvili-Vafa G_2 algebra [1] is a superconformal vertex algebra with six generators $\{L, G, \Phi, K, X, M\}$. It is an extension of the $N = 1$ superconformal algebra of central charge $c = 21/2$ (formed by the super-partners $\{L, G\}$) by two fields Φ and K , primary of conformal weight $\frac{3}{2}$ and 2 respectively, and their superpartners X and M (of conformal weight 2 and $\frac{5}{2}$ respectively). Their OPEs can be found in Appendix A in the language of lambda brackets of [2].

This superconformal algebra appeared as the chiral algebra associated to the sigma model with target a manifold with G_2 holonomy in [1] its classical counterpart had been studied by Howe and Papadopoulos in [3]. In fact this algebra is a member of a two-parameter family $SW(\frac{3}{2}, \frac{3}{2}, 2)$ previously studied in [4] where the author found the family of all superconformal algebras which are extension of the super-Virasoro algebra, i.e., the $N = 1$ superconformal algebra, by two primary supercurrents of conformal weights $\frac{3}{2}$ and 2 respectively. It is a family parametrized by (c, ε) (c is the central charge and ε the coupling constant) of non-linear W -algebras. Its generators and relations are recalled in Appendix B.

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The Shatashvili-Vafa G_2 algebra is a quotient of $SW(\frac{3}{2}, \frac{3}{2}, 2)$ with $c = \frac{21}{2}$ and $\varepsilon = 0$, in other words is the only one among this family which has central charge $c = \frac{21}{2}$ and contains the tri-critical Ising model as a subalgebra. It is precisely the fact that the Shatashvili-Vafa G_2 algebra appears as a W -algebra that motivated the authors to try to obtain this algebra as a quantum Hamiltonian reduction of some Lie superalgebra using the method developed in [7].

That $D(2, 1; \alpha)$ is the right Lie superalgebra candidate to be used in the Hamiltonian reduction is known from scattered results in the physics literature. It was shown in [10] that $SW(\frac{3}{2}, \frac{3}{2}, 2)$ is the symmetry algebra of the quantized Toda theory corresponding to $D(2, 1; \alpha)$ (in [9] was worked a classical version of this result in the case $\alpha = 1$ ($D(2, 1; \alpha) = osp(4|2)$)) and from the well established connection between the theory of nonlinear integrable equations and W -algebras, see for example [12].

A coset realization of the $SW(\frac{3}{2}, \frac{3}{2}, 2)$ superconformal algebra and therefore of the Shatashvili-Vafa algebra can be found in [14]. In [15] was shown that the Hamiltonian reduction of $D(2, 1; \alpha)$ coincides with this coset model (the authors however restrict their attention to the even part of the superalgebra).

Some representations of the Shatashvili-Vafa G_2 superconformal algebra can be found in [14], but the character formulae remains unknown. It was observed in [11] that in order to systematically study the representation theory and the character formula for this algebra one should construct the Shatashvili-Vafa algebra using the quantum Drinfeld-Sokolov reduction developed in [8, 16]. This step is accomplished in this article.

In section 2 we review how to perform the quantum Hamiltonian reduction of a Lie superalgebra as introduced in [7]. We recap some of the main theorems as well as under which conditions this Hamiltonian reduction process induces a free field realization.

In section 3 we prove that the $SW(\frac{3}{2}, \frac{3}{2}, 2)$ superconformal algebra is the quantum Hamiltonian reduction of the Lie superalgebra $D(2, 1; \alpha)$, and obtain a free field realization of the $SW(\frac{3}{2}, \frac{3}{2}, 2)$ algebra on a space of three free Bosons and three free Fermions. As particular cases $\alpha \in \{1, -\frac{1}{2}, -2\}$ we obtain the Shatashvili-Vafa G_2 algebra as a quantum Hamiltonian reduction of the Lie superalgebra $osp(4|2)$, and also the corresponding free field realizations. We summarize our main result as (see Theorem 3.1 its remark)

Theorem. Let \mathfrak{h} be the Cartan subalgebra of $D(2, 1; \alpha)$. It is a three dimensional vector space with a non-degenerate bilinear form $(,)$ given by the Cartan matrix. Consider $\Pi\mathfrak{h}^*$ the odd vector space (Π denotes parity change) with its natural bilinear form $-(,)$. Let $V_k(\mathfrak{h}_{\text{super}})$ be the super affine vertex algebra generated by three Bosons from \mathfrak{h} and three Fermions from $\Pi\mathfrak{h}^*$ and lambda brackets

$$[h_\lambda h'] = k\lambda(h, h'), \quad [\phi_\lambda \phi'] = -(\phi, \phi'), \quad h, h' \in \mathfrak{h}, \phi, \phi' \in \Pi\mathfrak{h}^*.$$

1. $SW(\frac{3}{2}, \frac{3}{2}, 2)$ is a sub-vertex algebra of $V_k(\mathfrak{h}_{\text{super}})$. The particular case of central charge $c = 21/2$ and vanishing coupling constant corresponds to

$k = -2/3$. In this case the fields G and Φ are given by (3.4) and all other fields can be obtained from these two.

2. The Shatashvili-Vafa G_2 superconformal algebra is a quotient of this algebra by an ideal generated in conformal weight $7/2$ (B.2).
3. For each α_i of the three odd simple roots of $D(2, 1, \alpha)$ there exists a module M_i of $V_k(\mathfrak{h}_{\text{super}})$ generated by a vector $|\alpha_i\rangle$ such that $h_n|\alpha_i\rangle = 0$ for $n > 0$, $h_0|\alpha_i\rangle = (h, \alpha_i)|\alpha_i\rangle$ and $\phi_n|\alpha_i\rangle = 0$ for $n > 0$ (for all $h \in \mathfrak{h}$ and $\phi \in \Pi\mathfrak{h}^*$). Let $\Gamma_i(z)$ be the unique intertwiner of type $\binom{V_k(\mathfrak{h}_{\text{super}})}{V_k(\mathfrak{h}_{\text{super}}) M_i}$ and $Q_i \in \text{Hom}(V_k(\mathfrak{h}_{\text{super}}), M_i)$ its zero mode. Then for generic values of (c, ε) we have $SW(\frac{3}{2}, \frac{3}{2}, 2) = \cap_i Q_i \subset V_k(\mathfrak{h}_{\text{super}})$.

In Section 3 the reader can find a stronger version of this Theorem as the generators for $SW(\frac{3}{2}, \frac{3}{2}, 2)$ are found for any values of the parameters (c, ε) .

2 Quantum reduction of Lie superalgebras

In this section we recall the construction of the W-algebras $W_k(\mathfrak{g}, x, f)$ introduced in [7]. We follow the presentation in [8].

To construct the vertex algebra $W_k(\mathfrak{g}, x, f)$ we need a quadruple (\mathfrak{g}, x, f, k) where $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a simple finite-dimensional Lie superalgebra with a non-degenerate even invariant supersymmetric bilinear form $(\cdot|\cdot)$, and $x, f \in \mathfrak{g}_0$ such that $\text{ad } x$ is diagonalizable on \mathfrak{g} with half-integer eigenvalues, $[x, f] = -f$, the eigenvalues of $\text{ad } x$ on the centralizer \mathfrak{g}^f of f in \mathfrak{g} are non-positive, and $k \in \mathbb{C}$.

We recall that a bilinear form $(\cdot|\cdot)$ on \mathfrak{g} is called even if $(\mathfrak{g}_0|\mathfrak{g}_1) = 0$, supersymmetric if $(\cdot|\cdot)$ is symmetric (resp. skewsymmetric) on \mathfrak{g}_0 (resp. \mathfrak{g}_1), invariant if $([a, b]|c) = (a|[b, c])$ for all $a, b, c \in \mathfrak{g}$.

A pair (x, f) satisfying the above properties can be obtained when x, f are part of an \mathfrak{sl}_2 triple, i.e., $[x, e] = e$, $[x, f] = -f$ and $[e, f] = x$. As this will be the case in the quantum reduction performed in section 3, we assume for the rest of this section that we are working with such a pair. Let $\mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j$ be the eigenspace decomposition with respect to $\text{ad } x$. Denote

$$\mathfrak{g}_+ = \bigoplus_{j>0} \mathfrak{g}_j, \quad \mathfrak{g}_- = \bigoplus_{j<0} \mathfrak{g}_j, \quad \mathfrak{g}_\leq = \mathfrak{g}_0 \bigoplus \mathfrak{g}_-.$$

Let $V_k(\mathfrak{g})$ denote the affine vertex algebra of level k associated to \mathfrak{g} . Denote by $F(A)$ the vertex algebra of free superfermions associated to a vector superspace A with an even skew-supersymmetric non-degenerate bilinear form $\langle \cdot | \cdot \rangle$, i.e., the λ -bracket is given by $[\varphi_\lambda \psi] = \langle \varphi | \psi \rangle$, $\varphi, \psi \in A$.

On the vector superspace $\mathfrak{g}_{1/2}$ the element f defines an even skew-supersymmetric non-degenerate bilinear form $\langle \cdot | \cdot \rangle_{ne}$ by the formula:

$$\langle a | b \rangle = (f|[a, b]).$$

The associated vertex algebra $F(\mathfrak{g}_{1/2})$ is called the vertex algebra of neutral free superfermions. Similary on the vector superspace $\Pi\mathfrak{g}_+ \oplus \Pi\mathfrak{g}_+^*$ (where Π denotes parity-reversing), define an even skew-supersymmetric non-degenerate bilinear form $\langle \cdot | \cdot \rangle_{ch}$ by:

$$\langle \Pi\mathfrak{g}_+ | \Pi\mathfrak{g}_+ \rangle_{ch} = 0 = \langle \Pi\mathfrak{g}_+^* | \Pi\mathfrak{g}_+^* \rangle_{ch},$$

$$\langle a | b^* \rangle_{ch} = -(-1)^{p(a)p(b^*)} \langle b^* | a \rangle_{ch} = b^*(a), \quad a \in \Pi\mathfrak{g}_+, b^* \in \Pi\mathfrak{g}_+^*,$$

where $p(a)$ denotes the parity of the element a . The associated vertex algebra $F(\Pi\mathfrak{g}_+ \oplus \Pi\mathfrak{g}_+^*)$ is called the vertex algebra of charged free superfermions. This vertex algebra carries an extra \mathbb{Z} -grading by charge by assigning: charge $\varphi = 1$ and charge $\varphi^* = -1$, $\varphi \in \Pi\mathfrak{g}_+$, $\varphi^* \in \Pi\mathfrak{g}_+^*$. Consider the vertex algebra

$$C(\mathfrak{g}, x, f, k) = V_k(\mathfrak{g}) \otimes F(\Pi\mathfrak{g}_+ \oplus \Pi\mathfrak{g}_+^*) \otimes F(\mathfrak{g}_{1/2}).$$

The charge decomposition of $F(\Pi\mathfrak{g}_+ \oplus \Pi\mathfrak{g}_+^*)$ induces a charge decomposition on $C(\mathfrak{g}, x, f, k)$ by declaring charge $V_k(\mathfrak{g}) = 0$ and charge $F(\mathfrak{g}_{1/2}) = 0$. This makes $C(\mathfrak{g}, x, f, k)$ a \mathbb{Z} -graded vertex algebra. We introduce a differential $d_{(0)}$ that makes $(C(\mathfrak{g}, x, f, k), d_{(0)})$ a \mathbb{Z} -graded complex as follows. Let $\{u_\alpha\}_{\alpha \in S_j}$ be a basis of each \mathfrak{g}_j , an let $S := \coprod_{j \in \frac{1}{2}\mathbb{Z}} S_j$, $S_+ = \coprod_{j > 0} S_j$. Put $m_\alpha := j$ if $\alpha \in S_j$. The structure constants $c_{\alpha\beta}^\gamma$ are defined by $[u_\alpha, u_\beta] = \sum_\gamma c_{\alpha\beta}^\gamma u_\gamma$ for $(\alpha, \beta, \gamma \in S)$. Denote by $\{\varphi_\alpha\}_{\alpha \in S_+}$ the corresponding basis of $\Pi\mathfrak{g}_+$ and by $\{\varphi^\alpha\}_{\alpha \in S_+}$ the basis of $\Pi\mathfrak{g}_+^*$ such that $\langle \varphi_\alpha, \varphi^\beta \rangle_{ch} = \delta_\alpha^\beta$. Similary denote by $\{\Phi_\alpha\}_{\alpha \in S_{1/2}}$ the corresponding basis of $\mathfrak{g}_{1/2}$, and by $\{\Phi^\alpha\}_{\alpha \in S_{1/2}}$ the dual basis with respect to $\langle \cdot, \cdot \rangle_{ne}$, i.e., $\langle \Phi_\alpha, \Phi^\beta \rangle_{ne} = \delta_\alpha^\beta$. It is useful to define Φ_u for any $u = \sum_{\alpha \in S} c_\alpha u_\alpha \in \mathfrak{g}$ by $\Phi_u := \sum_{\alpha \in S_{1/2}} c_\alpha \Phi_\alpha$. Define the odd field

$$\begin{aligned} d = & \sum_{\alpha \in S_+} (-1)^{p(u_\alpha)} : u_\alpha \varphi^\alpha : - \frac{1}{2} \sum_{\alpha, \beta, \gamma \in S_+} (-1)^{p(u_\alpha)p(u_\gamma)} c_{\alpha\beta}^\gamma : \varphi_\gamma \varphi^\alpha \varphi^\beta : \\ & + \sum_{\alpha \in S_+} (f|u_\alpha) \varphi^\alpha + \sum_{\alpha \in S_{1/2}} : \varphi^\alpha \Phi_\alpha : . \end{aligned}$$

Its Fourier mode $d_{(0)}$ is an odd derivation of all products of the vertex algebra $C(\mathfrak{g}, x, f, k)$, such that $d_{(0)}^2 = 0$ and that $d_{(0)}$ decreases the charge by 1. Thus $(C(\mathfrak{g}, x, f, k), d_{(0)})$ becomes a \mathbb{Z} -graded homology complex. Define the *affine W-algebra* $W_k(\mathfrak{g}, x, f)$ to be: as vector superspace the homology of this complex $W_k(\mathfrak{g}, x, f) := H(C(\mathfrak{g}, x, f, k), d_{(0)})$ together with the vertex algebra structure induced from $C(\mathfrak{g}, x, f, k)$. The vertex algebra $W_k(\mathfrak{g}, x, f)$ is also called the *quantum reduction* associated to the quadruple (\mathfrak{g}, x, f, k) . Define the Virasoro field of $C(\mathfrak{g}, x, f, k)$ by

$$L = L^{\mathfrak{g}} + \partial x + L^{ch} + L^{ne},$$

where

$$L^{\mathfrak{g}} = \frac{1}{2(k+h^\vee)} \sum_{\alpha \in S} (-1)^{p(u_\alpha)} : u_\alpha u^\alpha :,$$

is given by the Sugawara construction, where $\{u^\alpha\}_{\alpha \in S}$ is the dual basis to $\{u_\alpha\}_{\alpha \in S}$, i.e., $(u_\alpha | u^\beta) = \delta_\alpha^\beta$. Here we are assuming that $k \neq -h^\vee$, where h^\vee denotes the dual Coxeter number of \mathfrak{g} .

$$\begin{aligned} L^{ch} &= - \sum_{\alpha \in S_+} m_\alpha : \varphi^\alpha \partial \varphi_\alpha : + \sum_{\alpha \in S_+} (1 - m_\alpha) : (\partial \varphi^\alpha) \varphi_\alpha :, \\ L^{ne} &= \frac{1}{2} \sum_{\alpha \in S_{1/2}} : (\partial \Phi^\alpha) \Phi_\alpha :. \end{aligned}$$

The central charge of L is given by

$$\begin{aligned} c(\mathfrak{g}, x, f, k) &= \frac{k \dim \mathfrak{g}}{k + h^\vee} - 12k(x|x) \\ &\quad - \sum_{\alpha \in S_+} (-1)^{p(u_\alpha)} (12m_\alpha^2 - 12m_\alpha + 2) - \frac{1}{2} \dim \mathfrak{g}_{1/2}. \end{aligned} \quad (2.1)$$

With respect to L the fields a ($a \in \mathfrak{g}_j$), $\varphi_\alpha, \varphi^\alpha$ ($\alpha \in S_+$) and Φ_α ($\alpha \in S_{1/2}$) are primary vectors except for a ($a \in \mathfrak{g}_0$) such that $(a|x) \neq 0$, and the conformal weights are as follows: $\Delta(a) = 1 - j$ ($a \in \mathfrak{g}_j$), $\Delta(\varphi_\alpha) = 1 - m_\alpha$, $\Delta(\varphi^\alpha) = m_\alpha$ and $\Delta(\Phi_\alpha) = \frac{1}{2}$. In [7] is proved that $d_{(0)}L = 0$, then the homology class of L (which does not vanish) defines the Virasoro field of $W_k(\mathfrak{g}, x, f)$, which is again denoted by L .

To construct other fields of $W_k(\mathfrak{g}, x, f)$ define for each $v \in \mathfrak{g}_j$

$$J^{(v)} = v + \sum_{\alpha, \beta \in S_+} (-1)^{p(u_\alpha)} c_\beta^\alpha(v) : \varphi_\alpha \varphi^\beta :,$$

where the numbers $c_\beta^\alpha(v)$ are given by $[v, u_\alpha] = \sum_{\alpha \in S} c_\beta^\alpha(v) u_\alpha$. The field $J^{(v)} \in C(\mathfrak{g}, x, f, k)$ has the same charge, the same parity and the same conformal weight as the field v . The λ -bracket between these fields is as follows:

$$[J^{(v)}_\lambda J^{(v')}] = J^{([v, v'])} + \lambda (k(v|v') + \frac{1}{2} (\kappa_{\mathfrak{g}}(v, v') - \kappa_{\mathfrak{g}_0}(v, v'))), \quad (2.2)$$

if $v \in \mathfrak{g}_i, v' \in \mathfrak{g}_j$ and $ij \geq 0$ where $\kappa_{\mathfrak{g}}$ (resp. $\kappa_{\mathfrak{g}_0}$) denotes the Killing form on \mathfrak{g} (resp. \mathfrak{g}_0).

Denote by C^- the vertex subalgebra of the vertex algebra $C(\mathfrak{g}, x, f, k)$ generated by the fields $J^{(u)}$ for all $u \in \mathfrak{g}_\leq$, the fields φ^α for all $\alpha \in S_+$ and the fields Φ_α for all $\alpha \in S_{1/2}$. One of the main theorems on the structure of the vertex algebra $W_k(\mathfrak{g}, x, f)$ is the following:

Theorem 2.1. [8, Theorem 4.1] Let \mathfrak{g} be a simple finite-dimensional Lie superalgebra with an invariant bilinear form $(\cdot|\cdot)$ and let x, f be a pair of even elements of \mathfrak{g} such that $ad\ x$ is diagonalizable with eigenvalues in $\frac{1}{2}\mathbb{Z}$ and $[x, f] = -f$. Suppose that all eigenvalues of $ad\ x$ on g^f (the centralizer of f) are non-positive: $g^f = \oplus_{j \leq 0} g_j^f$. Then

- a) For each $a \in \mathfrak{g}_{-j}^f (j \geq 0)$ there exists a $d_{(0)}$ -closed field $J^{\{a\}}$ in C^- of conformal weight $1+j$ (with respect to L) such that $J^{\{a\}} - J^{(a)}$ is a linear combination of normal ordered products of the fields $J^{(b)}$, where $b \in \mathfrak{g}_{-s}$, $0 \leq s < j$, the fields Φ_α , where $\alpha \in S_{1/2}$, and the derivatives of these fields.
- b) The homology classes of the fields $J^{\{a_i\}}$, where a_1, a_2, \dots is a basis of g^f compatible with its $\frac{1}{2}\mathbb{Z}$ -gradation, strongly generate the vertex algebra $W_k(\mathfrak{g}, x, f)$.
- c) $H_0(C(\mathfrak{g}, x, f, k), d_{(0)}) = W_k(\mathfrak{g}, x, f)$ and $H_j(C(\mathfrak{g}, x, f, k), d_{(0)}) = 0$ if $j \neq 0$.

Remark 2.1. The complex $(C(\mathfrak{g}, x, f, k), d_{(0)})$ is formal, that is, the vertex algebra $W_k(\mathfrak{g}, x, f)$ is a subalgebra of $C(\mathfrak{g}, x, f, k)$ consisting of $d_{(0)}$ -closed charge 0 elements of C^- , furthermore the $J^{\{a\}}$ can be computed recursively, for example in the case $a \in g_{-1/2}^f$ the solution is unique and is given by:

Theorem 2.2. [8, Theorem 2.1 (d)]

For $v \in \mathfrak{g}_{-1/2}$ let

$$\begin{aligned} G^{\{v\}} &= J^{(v)} + \sum_{\beta \in S_{1/2}} : J^{([v, u_\beta])} \Phi^\beta : + \frac{(-1)^{p(v)+1}}{3} \sum_{\alpha, \beta \in S_{1/2}} : \Phi^\alpha \Phi^\beta \Phi_{[u_\beta[u_\alpha, v]]} : \\ &\quad - \sum_{\beta \in S_{1/2}} (k(v|u_\beta) + str_{g_+}(ad\ v)(ad\ u_\beta)) \partial \Phi^\beta, \end{aligned}$$

Then provided that $v \in \mathfrak{g}_{-1/2}^f$, we have $d_{(0)}(G^{\{v\}}) = 0$, hence the homology class of $G^{\{v\}}$ defines a field of the vertex algebra $W_k(\mathfrak{g}, x, f)$ of conformal weight $\frac{3}{2}$. This field is primary.

Remark 2.2. In the case $\mathfrak{g}^f \subset g_{\leq}$ Theorem 2.1 and the identity (2.2) provides a construction of the vertex algebra $W_k(\mathfrak{g}, x, f)$ as a subalgebra of $V_{\nu_k}(\mathfrak{g}_{\leq}) \otimes F(\mathfrak{g}_{1/2})$ where ν_k is the 2-cocycle on $\mathfrak{g}_{\leq}[t, t^{-1}]$ given by

$$\nu_k(at^m, bt^n) = m\delta_{m,-n} (k(a|b) + \frac{1}{2} (\kappa_{\mathfrak{g}}(a, b) - \kappa_{\mathfrak{g}_0}(a, b))) \quad (2.3)$$

for $a, b \in \mathfrak{g}_{\leq}$ and $m, n \in \mathbb{Z}$.

Remark 2.3. Furthermore if this 2-cocycle is trivial outside $\mathfrak{g}_0[t, t^{-1}]$, the canonical homomorphism $\mathfrak{g}_{\leq} \rightarrow \mathfrak{g}_0$ induces a homomorphism from $V_{\nu_k}(\mathfrak{g}_{\leq}) \otimes F(\mathfrak{g}_{1/2})$ to $V_{\nu_k}(\mathfrak{g}_0) \otimes F(\mathfrak{g}_{1/2})$, obtaining in this way a free field realization of $W_k(\mathfrak{g}, x, f)$ inside $V_{\nu_k}(\mathfrak{g}_0) \otimes F(\mathfrak{g}_{1/2})$.

3 Quantum Hamiltonian Reduction of $D(2, 1; \alpha)$

In this section we prove that the family $SW(\frac{3}{2}, \frac{3}{2}, 2)$ of W-algebras which has generators $\{G, H, L, \tilde{M}, W, U\}$ of conformal weights $(\frac{3}{2}, \frac{3}{2}, 2, 2, 2, \frac{5}{2})$ and relations as given in Appendix B can be obtained as the quantum Hamiltonian reduction of $D(2, 1; \alpha)$. As a corollary we obtain a free field realization of this family. As a particular case we obtain a free-field realization of the Shatashvili-Vafa G_2 algebra on a space of three free Bosons and three free Fermions.

The Lie superalgebra $D(2, 1; \alpha)$ where $\alpha \in \mathbb{C} \setminus \{-1, 0\}$ is a one-parameter family of exceptional Lie superalgebras of rank 3 and dimension 17, which contains $D(2, 1) = osp(4, 2)$ as special cases (when $\alpha \in \{1, -\frac{1}{2}, -2\}$), see [13].

We present $\mathfrak{g} = D(2, 1; \alpha)$ as the contragradient Lie superalgebra associated to the Cartan matrix $A = (a_{ij})_{i,j}$ and $\tau = \{1, 2, 3\}$

$$(a_{ij})_{i,j=1}^3 = \begin{pmatrix} 0 & 1 & \alpha \\ 1 & 0 & -1 - \alpha \\ \alpha & -1 - \alpha & 0 \end{pmatrix}. \quad (3.1)$$

We have generators $\{h_1, h_2, h_3, e_1, e_2, e_3, f_1, f_2, f_3\}$, h_i being even for all i and e_i, f_i being odd for all i and relations

$$[e_i, f_j] = \delta_{ij} h_i, \quad [h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j.$$

Introduce the elements:

$$[e_1, e_2] =: e_{12}, \quad [e_1, e_3] =: e_{13}, \quad [e_2, e_3] =: e_{23}, \quad [e_1, e_{23}] =: e_{123},$$

$$[f_1, f_2] =: f_{12}, \quad [f_1, f_3] =: f_{13}, \quad [f_2, f_3] =: f_{23}, \quad [f_1, f_{23}] =: f_{123}.$$

Recall that \mathfrak{g} has vanishing Killing form and consequently the dual Coxeter number $h^\vee = 0$. Fix the following non-degenerate even supersymmetric invariant bilinear form (\cdot, \cdot)

$$\begin{aligned} (h_i, h_j) &= a_{ij}, \quad (e_i, f_j) = \delta_{ij}, \quad (e_{12}, f_{12}) = (f_{12}, e_{12}) = -1, \\ (e_{13}, f_{13}) &= (f_{13}, e_{13}) = -\alpha, \quad (e_{23}, f_{23}) = (f_{23}, e_{23}) = 1 + \alpha, \\ (e_{123}, f_{123}) &= -(f_{123}, e_{123}) = (1 + \alpha)^2. \end{aligned}$$

To perform the quantum Hamiltonian reduction we take the pair (x, f) :

$$x := \frac{(\alpha+1)}{2\alpha} h_1 + \frac{\alpha}{2(\alpha+1)} h_2 + \frac{1}{2\alpha(\alpha+1)} h_3, \quad f := f_{12} + f_{13} + f_{23}.$$

This pair together with $e = (-\frac{1}{2})e_{12} + (-\frac{1}{2\alpha^2})e_{13} + (-\frac{1}{2(\alpha+1)^2})e_{23}$ forms an sl_2 triple. We have the following eigenspace decomposition of the algebra with

respect to $ad\ x$:

$$\begin{array}{cccccc}
\mathfrak{g}_{-3/2} & \mathfrak{g}_{-1} & \mathfrak{g}_{-1/2} & \mathfrak{g}_0 & \mathfrak{g}_{1/2} & \mathfrak{g}_1 & \mathfrak{g}_{3/2} \\
f_{123} & f_{12} & f_1 & h_1 & e_1 & e_{12} & e_{123} \\
& f_{13} & f_2 & h_2 & e_2 & e_{13} & \\
& f_{23} & f_3 & h_3 & e_3 & e_{23} &
\end{array}$$

Furthermore $\mathfrak{g}^f = \mathfrak{g}_{-1/2}^f \oplus \mathfrak{g}_{-1}^f \oplus \mathfrak{g}_{-3/2}^f$ with $\dim \mathfrak{g}_{-1/2}^f = 2$, $\dim \mathfrak{g}_{-1}^f = 3$ and $\dim \mathfrak{g}_{-3/2}^f = 1$. This shows that the algebra $W_k(\mathfrak{g}, x, f)$ has six generators with the expected conformal weights.

The set of vectors $\{e_1, e_2, e_3\}$ is a basis of $\mathfrak{g}_{1/2}$, denote by $\Phi_1 := e_1$, $\Phi_2 := e_2$ and $\Phi_3 := e_3$ the corresponding free neutral Fermions. The non-zero values of the (symmetric) bilinear form $\langle \cdot | \cdot \rangle_{ne}$ on $\mathfrak{g}_{1/2}$ are given by:

$$\langle \Phi_1 | \Phi_2 \rangle_{ne} = -1, \quad \langle \Phi_1 | \Phi_3 \rangle_{ne} = -\alpha, \quad \langle \Phi_2 | \Phi_3 \rangle_{ne} = 1 + \alpha,$$

(note that this is exactly minus the Cartan matrix of $D(2, 1; \alpha)$.) Then the free neutral fermions satisfy the following non-zero λ -brackets:

$$[\Phi_1 \lambda \Phi_2] = -1, \quad [\Phi_1 \lambda \Phi_3] = -\alpha, \quad [\Phi_2 \lambda \Phi_3] = 1 + \alpha,$$

and the dual free neutral fermions with respect to $\langle \cdot | \cdot \rangle_{ne}$ are:

$$\begin{aligned}
\Phi^1 &= \left(-\frac{1+\alpha}{2\alpha}\right)\Phi_1 + \left(-\frac{1}{2}\right)\Phi_2 + \left(-\frac{1}{2\alpha}\right)\Phi_3, \\
\Phi^2 &= \left(-\frac{1}{2}\right)\Phi_1 + \left(-\frac{\alpha}{2+2\alpha}\right)\Phi_2 + \left(\frac{1}{2+2\alpha}\right)\Phi_3, \\
\Phi^3 &= \left(-\frac{1}{2\alpha}\right)\Phi_1 + \left(\frac{1}{2+2\alpha}\right)\Phi_2 + \left(-\frac{1}{2\alpha+2\alpha^2}\right)\Phi_3.
\end{aligned}$$

We fix the basis $\{h_1, h_2, h_3, f_1, f_2, f_3, f_{12}, f_{13}, f_{23}, f_{123}\}$ of \mathfrak{g}_{\leq} compatible with the $\frac{1}{2}\mathbb{Z}$ and \mathbb{Z}_2 gradation of \mathfrak{g} . We consider the building blocks $J^{(v)}$ for each v that belongs to the above basis, (2.2) reduces to

$$[J^{(v)} \lambda J^{(v')}] = J^{([v, v'])} + \lambda k(v|v'),$$

because the Killing form $\kappa_{\mathfrak{g}}$ of \mathfrak{g} is zero and \mathfrak{g}_0 equals the Cartan subalgebra \mathfrak{h} of \mathfrak{g} , that is, the generators $J^{(v)}$ obey the same commutation relations as the generators of $V_k(D(2, 1; \alpha))$. Using Remark 2.2 we obtain that $W_k(\mathfrak{g}, x, f)$ is a subalgebra of $V_k(\mathfrak{g}_{\leq}) \otimes F(\mathfrak{g}_{1/2})$. For this reason and to simplify the notation we denote $J^{(v)}$ simply by v . Furthermore as the cocycle (2.3) is the original cocycle of $V_k(D(2, 1; \alpha))$ and this cocycle is trivial in \mathfrak{g}_{\leq} outside $\mathfrak{g}_0 = \mathfrak{h}$, Remark 2.3 gives a free field realization of $W_k(\mathfrak{g}, x, f)$ inside $V_k(\mathfrak{h}) \otimes F(\mathfrak{g}_{1/2})$.

Let $J^{\{f_i\}}$ denote the d_0 -closed fields associated to $\{f_i\}_{i=1}^3$ provided by Theorem 2.1. Using Theorem 2.2 we can compute $J^{\{f_i\}}$ explicitly:

$$\begin{aligned}
J^{\{f_1\}} &= f_1 + \left(\frac{\alpha^2-1}{3}\right) : \Phi^1 \Phi^2 \Phi^3 : + : \Phi^1 h_1 : + k \partial \Phi^1, \\
J^{\{f_2\}} &= f_2 + \left(\frac{\alpha(\alpha+2)}{3}\right) : \Phi^1 \Phi^2 \Phi^3 : + : \Phi^2 h_2 : + k \partial \Phi^2, \\
J^{\{f_3\}} &= f_3 + \left(\frac{2\alpha+1}{3}\right) : \Phi^1 \Phi^2 \Phi^3 : + : \Phi^3 h_3 : + k \partial \Phi^3.
\end{aligned}$$

We can compute the other fields $J^{\{f_{1,2}\}}, J^{\{f_{1,3}\}}, J^{\{f_{2,3}\}}, J^{\{f_{1,2,3}\}}$ given by Theorem 2.1 that jointly with $\{J^{\{f_i\}}\}_{i=1}^3$ strongly generate $W_k(\mathfrak{g}, x, f)$, but in the $SW(\frac{3}{2}, \frac{3}{2}, 2)$ superconformal algebra we can recover (using λ -brackets) all the fields from the generators in conformal weight $\frac{3}{2}$, i.e., G and H (see Appendix B). Thus we only need to construct G and H from $\{J^{\{f_i\}}\}_{i=1}^3$.

In order to do that observe that:

$$a_1 f_1 + a_2 f_2 + a_3 f_3 \in \mathfrak{g}_{-1/2}^f \Leftrightarrow a_1 + a_2(-\frac{\alpha}{\alpha+1}) + a_3(-\frac{1}{\alpha+1}) = 0, \quad (3.2)$$

and that the central charge of the Virasoro field of $W_k(\mathfrak{g}, x, f)$ given by formula (2.1) is $c(\alpha, k) = \frac{9}{2} - 12k(x|x) = \frac{9}{2} - \frac{6k(1+\alpha+\alpha^2)}{\alpha(1+\alpha)}$.

We want to define a field G such that $\{G, L := \frac{1}{2}G_{(0)}G\}$ generate an $N = 1$ superconformal algebra with the above central charge, this is accomplished taking $a_1 = a_2 = a_3 = \frac{i}{\sqrt{k}}$, i.e.,

$$G := \frac{i}{\sqrt{k}} \left(J^{\{f_1\}} + J^{\{f_2\}} + J^{\{f_3\}} \right).$$

We are looking for a vector H of conformal weight $\frac{3}{2}$, such that:

$$G_{(j)}H = 0, \quad j > 0, \quad (3.3)$$

The most general vector of conformal weight $\frac{3}{2}$ given by (3.2) is

$$\left(\frac{\alpha}{\alpha+1}a_2 + \frac{1}{\alpha+1}a_3 \right) J^{\{f_1\}} + a_2 J^{\{f_2\}} + a_3 J^{\{f_3\}},$$

(3.3) imposes the condition $a_2\alpha(-1+2k)(1+2\alpha) + a_3(2k-\alpha)(2+\alpha) = 0$, which has as solution

$$\begin{aligned} a_1' &:= \alpha(-1+\alpha)(1+2k+\alpha), \\ a_2' &:= (-1)(2k-\alpha)(2+\alpha)(1+\alpha), \\ a_3' &:= \alpha(-1+2k)(1+2\alpha)(1+\alpha). \end{aligned}$$

It follows from $H_{(2)}H = \frac{2c}{3}$ (cf. (B.1)) that we need to rescale this solution to define $H = \sum_{i=1}^3 a_i J^{\{f_i\}}$ with

$$a_i := \left(-\frac{3}{2}(-1+2k)\alpha^2(1+\alpha)^2 (2k+4k^2-\alpha(1+\alpha)) \right)^{-\frac{1}{2}} a_i'.$$

We can obtain all other generators from G and H , to perform this computations we use Thielemans's software [18]. Listed below are the explicit expressions of all the generators of $W_k(\mathfrak{g}, x, f)$ as a subalgebra of $V_k(\mathfrak{g}_{\leq}) \otimes F(\mathfrak{g}_{1/2})$:

$$\begin{aligned} G &= \frac{i}{\sqrt{k}}f_1 + \frac{i}{\sqrt{k}}f_2 + \frac{i}{\sqrt{k}}f_3 + \frac{i}{\sqrt{k}} : \Phi^1 h_1 : + \frac{i}{\sqrt{k}} : \Phi^2 h_2 : + \frac{i}{\sqrt{k}} : \Phi^3 h_3 : \\ &\quad + i\sqrt{k}\partial\Phi^1 + i\sqrt{k}\partial\Phi^2 + i\sqrt{k}\partial\Phi^3, \end{aligned}$$

$$\begin{aligned}
L = & -\frac{1}{k}f_{12} - \frac{1}{k}f_{13} - \frac{1}{k}f_{23} + \frac{(1+\alpha)}{4k\alpha} : h_1h_1 : + \frac{1}{2k} : h_1h_2 : + \frac{1}{2k\alpha} : h_1h_3 : \\
& + \frac{\alpha}{4k+4k\alpha} : h_2h_2 : - \frac{1}{2k+2k\alpha} : h_2h_3 : + \frac{1}{4k\alpha+4k\alpha^2} : h_3h_3 : + \frac{1}{k} : \Phi^1f_2 : \\
& + \frac{\alpha}{k} : \Phi^1f_3 : + \frac{1}{2} : \Phi^1\partial\Phi^2 : + \frac{1}{2}\alpha : \Phi^1\partial\Phi^3 : + \frac{1}{k} : \Phi^2f_1 : - \frac{(1+\alpha)}{k} : \Phi^2f_3 : \\
& + \frac{1}{2}(-1-\alpha) : \Phi^2\partial\Phi^3 : + \frac{\alpha}{k} : \Phi^3f_1 : - \frac{(1+\alpha)}{k} : \Phi^3f_2 : - \frac{1}{2} : \partial\Phi^1\Phi^2 : \\
& - \frac{1}{2}\alpha : \partial\Phi^1\Phi^3 : + \frac{1}{2}(1+\alpha) : \partial\Phi^2\Phi^3 : + \frac{(1+\alpha)}{2\alpha}\partial h_1 + \frac{\alpha}{2+2\alpha}\partial h_2 \\
& + \frac{1}{2\alpha+2\alpha^2}\partial h_3,
\end{aligned}$$

$$\begin{aligned}
H = & \frac{1}{\sqrt{-\frac{3}{2}(-1+2k)\alpha^2(1+\alpha)^2(2k+4k^2-\alpha(1+\alpha))}} \left((-1+\alpha)\alpha(1+2k+\alpha)(f_1+ : \Phi^1h_1 : \right. \\
& + k\partial\Phi^1) - (2k-\alpha)(2+3\alpha+\alpha^2)(f_2+ : \Phi^2h_2 : + k\partial\Phi^2) \\
& + (-1+2k)\alpha(1+3\alpha+2\alpha^2)(f_3+ : \Phi^3h_3 : + k\partial\Phi^3) \\
& \left. + \alpha(1+\alpha)(-3\alpha(1+\alpha)+4k(1+\alpha+\alpha^2)) : \Phi^1\Phi^2\Phi^3 : \right),
\end{aligned}$$

$$\begin{aligned}
\tilde{M} = & \frac{1}{\sqrt{-\frac{3}{2}(-1+2k)\alpha^2(1+\alpha)^2(2k+4k^2-\alpha(1+\alpha))}} \left(\frac{i(1+2\alpha)(-4k+\alpha+\alpha^2)}{\sqrt{k}} (f_{12} - \frac{1}{2} : h_1h_2 : \right. \\
& - : \Phi^1f_2 - : \Phi^2f_1 :) + \frac{i(2+\alpha)(-1+(-1+4k)\alpha)}{\sqrt{k}} (\alpha f_{13} - \frac{1}{2} : h_1h_3 : - \alpha^2 : \Phi^1f_3 : \\
& - \alpha^2 : \Phi^3f_1 :) + \frac{i(-1+\alpha^2)(-\alpha+4k(1+\alpha))}{\sqrt{k}} \left(f_{23} + \frac{1}{2(1+\alpha)} : h_2h_3 : \right. \\
& + (1+\alpha) : \Phi^3f_2 : + (1+\alpha) : \Phi^2f_3 :) \\
& - i\sqrt{k}(-1+\alpha)(1+\alpha)(1+2k+\alpha) (\partial h_1 + \frac{1}{2k} : h_1h_1 :) \\
& + i\sqrt{k}(2k-\alpha)\alpha(2+\alpha) (\partial h_2 + \frac{1}{2k} : h_2h_2 :) \\
& - i\sqrt{k}(-1+2k)(1+2\alpha) (\partial h_3 + \frac{1}{2k} : h_3h_3 :) \\
& + \frac{i(-3\alpha(1+\alpha)+4k(1+\alpha+\alpha^2))}{2\sqrt{k}} \left(-(1+\alpha) : \Phi^1\Phi^2h_1 : + \alpha : \Phi^1\Phi^2h_2 : - : \Phi^1\Phi^2h_3 : \right. \\
& + \alpha(1+\alpha) : \Phi^1\Phi^3h_1 : + \alpha^2 : \Phi^1\Phi^3h_2 : - \alpha : \Phi^1\Phi^3h_3 : - (1+\alpha)^2 : \Phi^2\Phi^3h_1 : \\
& - \alpha(1+\alpha) : \Phi^2\Phi^3h_2 : - (1+\alpha) : \Phi^2\Phi^3h_3 :) \\
& - i\sqrt{k}(-1+\alpha)\alpha(1+2k+\alpha) : \Phi^1\partial\Phi^2 : \\
& - i\sqrt{k}(-1+\alpha)\alpha^2(1+2k+\alpha) : \Phi^1\partial\Phi^3 : \\
& - i\sqrt{k}(2k-\alpha)(1+\alpha)^2(2+\alpha) : \Phi^2\partial\Phi^3 : \\
& - i\sqrt{k}(2k-\alpha)(2+3\alpha+\alpha^2) : \partial\Phi^1\Phi^2 : \\
& + i\sqrt{k}(-1+2k)\alpha^2(1+3\alpha+2\alpha^2) : \partial\Phi^1\Phi^3 : \\
& \left. - i\sqrt{k}(-1+2k)\alpha(1+\alpha)^2(1+2\alpha) : \partial\Phi^2\Phi^3 : \right),
\end{aligned}$$

$$\begin{aligned}
W = & \frac{\mu}{(-3\alpha(1+\alpha)+4k(1+\alpha+\alpha^2))} \left(\frac{(2k+\alpha+\alpha^2)}{k} (-f_{12} + \frac{1}{2} : h_1 h_2 : + : \Phi^1 f_2 : \right. \\
& + : \Phi^2 f_1 :) + \frac{(1+\alpha+2k\alpha)}{k} (-\alpha f_{13} + \frac{1}{2} : h_1 h_3 : + \alpha^2 : \Phi^1 f_3 : + \alpha^2 : \Phi^3 f_1 :) \\
& + \frac{(1+\alpha)(\alpha+2k(1+\alpha))}{k} (-f_{23} - (1+\alpha) : \Phi^2 f_3 : - (1+\alpha) : \Phi^3 f_2 :) \\
& + \frac{(1+\alpha)(1+2k+\alpha)}{4k} : h_1 h_1 : + \frac{\alpha(-2k+\alpha)}{4k} : h_2 h_2 : + \frac{-2k+\alpha(-2k-1)}{2k} : h_2 h_3 : \\
& + \frac{-2k+1}{4k} : h_3 h_3 : + (-1+\alpha^2) : \Phi^1 \Phi^2 h_1 : - \alpha(2+\alpha) : \Phi^1 \Phi^2 h_2 : \\
& + (-1-2\alpha) : \Phi^1 \Phi^2 h_3 : + (\alpha-\alpha^3) : \Phi^1 \Phi^3 h_1 : - \alpha^2(2+\alpha) : \Phi^1 \Phi^3 h_2 : \\
& - \alpha(1+2\alpha) : \Phi^1 \Phi^3 h_3 : - \alpha(1+2k+\alpha) : \Phi^1 \partial \Phi^2 : \\
& - \alpha^2(1+2k+\alpha) : \Phi^1 \partial \Phi^3 : + (-1+\alpha)(1+\alpha)^2 : \Phi^2 \Phi^3 h_1 : \\
& + \alpha(2+3\alpha+\alpha^2) : \Phi^2 \Phi^3 h_2 : + (-1-3\alpha-2\alpha^2) : \Phi^2 \Phi^3 h_3 : \\
& - (2k-\alpha)(1+\alpha)^2 : \Phi^2 \partial \Phi^3 : - (2k-\alpha)(1+\alpha) : \partial \Phi^1 \Phi^2 : \\
& - (-1+2k)\alpha^2(1+\alpha) : \partial \Phi^1 \Phi^3 : + (-1+2k)\alpha(1+\alpha)^2 : \partial \Phi^2 \Phi^3 : \\
& + \frac{1}{2}(1+\alpha)(1+2k+\alpha)\partial h_1 + \frac{1}{2}\alpha(-2k+\alpha)\partial h_2 + \frac{1-2k}{2}\partial h_3 \Big),
\end{aligned}$$

$$\begin{aligned}
U = & \frac{\mu}{(-3\alpha(1+\alpha)+4k(1+\alpha+\alpha^2))} \left(-\frac{6i\alpha}{\sqrt{k}} f_{123} + \frac{3i(1+\alpha)}{\sqrt{k}} : h_1 f_2 : + \frac{3i\alpha(1+\alpha)}{\sqrt{k}} : h_1 f_3 : \right. \\
& - \frac{3i\alpha}{\sqrt{k}} : h_2 f_1 : + \frac{3i\alpha(1+\alpha)}{\sqrt{k}} : h_2 f_3 : - \frac{3i\alpha}{\sqrt{k}} : h_3 f_1 : \\
& + \frac{3i(1+\alpha)}{\sqrt{k}} : h_3 f_2 : + \frac{6i\alpha(1+\alpha)}{\sqrt{k}} : \Phi^1 f_{23} : - \frac{3i\alpha}{\sqrt{k}} : \Phi^1 h_1 h_2 : \\
& - \frac{3i\alpha}{\sqrt{k}} : \Phi^1 h_1 h_3 : - \frac{3i\alpha(1+\alpha)}{\sqrt{k}} : \Phi^1 \Phi^2 f_1 : + \frac{3i\alpha(1+\alpha)}{\sqrt{k}} : \Phi^1 \Phi^2 f_2 : \\
& + \frac{3i\alpha(1+3\alpha+2\alpha^2)}{\sqrt{k}} : \Phi^1 \Phi^2 f_3 : + i\sqrt{k}\alpha(1+3\alpha+2\alpha^2) : \Phi^1 \Phi^2 \partial \Phi^3 : \\
& - \frac{3i\alpha^2(1+\alpha)}{\sqrt{k}} : \Phi^1 \Phi^3 f_1 : + \frac{3i\alpha(2+3\alpha+\alpha^2)}{\sqrt{k}} : \Phi^1 \Phi^3 f_2 : + \frac{3i\alpha^2(1+\alpha)}{\sqrt{k}} : \Phi^1 \Phi^3 f_3 : \\
& - 3i\sqrt{k}\alpha(1+\alpha) : \Phi^1 \partial \Phi^2 \Phi^2 : - i\sqrt{k}\alpha(2+3\alpha+\alpha^2) : \Phi^1 \partial \Phi^2 \Phi^3 : \\
& - 3i\sqrt{k}\alpha^2(1+\alpha) : \Phi^1 \partial \Phi^3 \Phi^3 : - \frac{i\alpha(1+2k+\alpha)}{\sqrt{k}} : \Phi^1 \partial h_1 : \\
& + \frac{6i\alpha(1+\alpha)}{\sqrt{k}} : \Phi^2 f_{13} : + \frac{3i(1+\alpha)}{\sqrt{k}} : \Phi^2 h_1 h_2 : + \frac{3i(1+\alpha)}{\sqrt{k}} : \Phi^2 h_2 h_3 : \\
& - \frac{3i\alpha(-1+\alpha^2)}{\sqrt{k}} : \Phi^2 \Phi^3 f_1 : + \frac{3i\alpha(1+\alpha)^2}{\sqrt{k}} : \Phi^2 \Phi^3 f_2 : - \frac{3i\alpha(1+\alpha)^2}{\sqrt{k}} : \Phi^2 \Phi^3 f_3 : \\
& + 3i\sqrt{k}\alpha(1+\alpha)^2 : \Phi^2 \partial \Phi^3 \Phi^3 : + \frac{i(2k-\alpha)(1+\alpha)}{\sqrt{k}} : \Phi^2 \partial h_2 : + \frac{6i\alpha(1+\alpha)}{\sqrt{k}} : \Phi^3 f_{12} : \\
& + \frac{3i\alpha(1+\alpha)}{\sqrt{k}} : \Phi^3 h_1 h_3 : + \frac{3i\alpha(1+\alpha)}{\sqrt{k}} : \Phi^3 h_2 h_3 : + \frac{i(-1+2k)\alpha(1+\alpha)}{\sqrt{k}} : \Phi^3 \partial h_3 : \\
& - \frac{2i(-1+k-\alpha)\alpha}{\sqrt{k}} : \partial \Phi^1 h_1 : - 3i\sqrt{k}\alpha : \partial \Phi^1 h_2 : - 3i\sqrt{k}\alpha : \partial \Phi^1 h_3 : \\
& - 3i\sqrt{k}\alpha(1+\alpha) : \partial \Phi^1 \Phi^1 \Phi^2 : - 3i\sqrt{k}\alpha^2(1+\alpha) : \partial \Phi^1 \Phi^1 \Phi^3 : \\
& - i\sqrt{k}\alpha(-1+\alpha^2) : \partial \Phi^1 \Phi^2 \Phi^3 : + 3i\sqrt{k}(1+\alpha) : \partial \Phi^2 h_1 : \\
& + \frac{2i(1+\alpha)(k+\alpha)}{\sqrt{k}} : \partial \Phi^2 h_2 : + 3i\sqrt{k}(1+\alpha) : \partial \Phi^2 h_3 : \\
& + 3i\sqrt{k}\alpha(1+\alpha)^2 : \partial \Phi^2 \Phi^2 \Phi^3 : + 3i\sqrt{k}\alpha(1+\alpha) : \partial \Phi^3 h_1 :
\end{aligned}$$

$$\begin{aligned}
& +3i\sqrt{k}\alpha(1+\alpha) : \partial\Phi^3 h_2 : + \frac{2i(1+k)\alpha(1+\alpha)}{\sqrt{k}} : \partial\Phi^3 h_3 : \\
& - \frac{i\alpha(1+2k+\alpha)}{\sqrt{k}} \partial f_1 + \frac{i(2k-\alpha)(1+\alpha)}{\sqrt{k}} \partial f_2 + \frac{i(-1+2k)\alpha(1+\alpha)}{\sqrt{k}} \partial f_3 \\
& - \frac{1}{2}i\sqrt{k}(-1+4k-\alpha)\alpha\partial^2\Phi^1 + \frac{1}{2}i\sqrt{k}(1+\alpha)(4k+\alpha)\partial^2\Phi^2 \\
& + \frac{1}{2}i\sqrt{k}(1+4k)\alpha(1+\alpha)\partial^2\Phi^3 \Big),
\end{aligned}$$

where $\mu = \sqrt{\frac{9c(4+\varepsilon^2)}{2(27-2c)}}$ and $\varepsilon(\alpha, k) = -\frac{4i\sqrt{\frac{2}{3}}k^{3/2}(1+2\alpha)(-2+\alpha+\alpha^2)}{3\sqrt{-(-1+2k)\alpha^2(1+\alpha)^2(2k+4k^2-\alpha(1+\alpha))}}$.

One can check straightforwardly with the aid of [18] that the λ -brackets of the algebra $W_k(\mathfrak{g}, x, f)$ coincides with the λ -brackets of the family of superconformal algebras $SW(\frac{3}{2}, \frac{3}{2}, 2)$ with parameters $(c(\alpha, k), \varepsilon(\alpha, k))$. Shatashvili-Vafa's G_2 superconformal algebra is a quotient of this algebra for $(c, \varepsilon) = (21/2, 0)$ modulo an ideal generated in conformal weight $\frac{7}{2}$ (cf. Remark B.1), in particular, the explicit commutation relations obtained in [1] are an artifact of the free field realization the authors used [5]. Solving $(c, \varepsilon) = (21/2, 0)$ in terms of α and k there are three solutions: $\{\alpha = 1, k = -2/3\}$, $\{\alpha = -2, k = -2/3\}$ and $\{\alpha = -1/2, k = 1/3\}$. Precisely for this values of α the superalgebra $D(2, 1; \alpha)$ is nothing but the superalgebra $osp(4|2)$, then the Shatashvili-Vafa G_2 superconformal algebra is a quotient of the quantum Hamiltonian reduction of $osp(4|2)$.

Remark 3.1. The existence of the ideal (B.2) can be guessed from the fact that the affine vertex algebra $V_k(osp(4|2))$ at level $k \in \{-\frac{2}{3}, \frac{1}{3}\}$ is not simple, i.e., contains a non-trivial ideal [17].

Listed below are the explicit expressions of all the generators of the Shatashvili-Vafa G_2 superconformal algebra in the case $\{\alpha = 1, k = -2/3\}$. Note that we are using the change of basis (B.3).

$$\begin{aligned}
G &= \sqrt{\frac{3}{2}}f_1 + \sqrt{\frac{3}{2}}f_2 + \sqrt{\frac{3}{2}}f_3 + \sqrt{\frac{3}{2}} : \Phi^1 h_1 : + \sqrt{\frac{3}{2}} : \Phi^2 h_2 : \\
&+ \sqrt{\frac{3}{2}} : \Phi^3 h_3 : - \sqrt{\frac{2}{3}}\partial\Phi^1 - \sqrt{\frac{2}{3}}\partial\Phi^2 - \sqrt{\frac{2}{3}}\partial\Phi^3,
\end{aligned}$$

$$\begin{aligned}
L &= \frac{3}{2}f_{12} + \frac{3}{2}f_{13} + \frac{3}{2}f_{23} - \frac{3}{4} : h_1 h_1 : - \frac{3}{4} : h_1 h_2 : - \frac{3}{4} : h_1 h_3 : - \frac{3}{16} : h_2 h_2 : \\
&+ \frac{3}{8} : h_2 h_3 : - \frac{3}{16} : h_3 h_3 : - \frac{3}{2} : \Phi^1 f_2 : - \frac{3}{2} : \Phi^1 f_3 : + \frac{1}{2} : \Phi^1 \partial\Phi^2 : \\
&+ \frac{1}{2} : \Phi^1 \partial\Phi^3 : - \frac{3}{2} : \Phi^2 f_1 : + 3 : \Phi^2 f_3 : - : \Phi^2 \partial\Phi^3 : - \frac{3}{2} : \Phi^3 f_1 : \\
&+ 3 : \Phi^3 f_2 : - \frac{1}{2} : \partial\Phi^1 \Phi^2 : - \frac{1}{2} : \partial\Phi^1 \Phi^3 : + : \partial\Phi^2 \Phi^3 : + \partial h_1 \\
&+ \frac{1}{4}\partial h_2 + \frac{1}{4}\partial h_3,
\end{aligned}$$

$$\Phi = 3f_2 - 3f_3 - 6 : \Phi^1 \Phi^2 \Phi^3 : + 3 : \Phi^2 h_2 : - 3 : \Phi^3 h_3 : - 2\partial\Phi^2 + 2\partial\Phi^3,$$

$$\begin{aligned}
K = & 3\sqrt{\frac{3}{2}}f_{12} - 3\sqrt{\frac{3}{2}}f_{13} - \frac{3}{2}\sqrt{\frac{3}{2}} : h_1 h_2 : + \frac{3}{2}\sqrt{\frac{3}{2}} : h_1 h_3 : - \frac{3}{4}\sqrt{\frac{3}{2}} : h_2 h_2 : \\
& + \frac{3}{4}\sqrt{\frac{3}{2}} : h_3 h_3 : - 3\sqrt{\frac{3}{2}} : \Phi^1 f_2 : + 3\sqrt{\frac{3}{2}} : \Phi^1 f_3 : + 3\sqrt{\frac{3}{2}} : \Phi^1 \Phi^2 h_1 : \\
& - \frac{3}{2}\sqrt{\frac{3}{2}} : \Phi^1 \Phi^2 h_2 : + \frac{3}{2}\sqrt{\frac{3}{2}} : \Phi^1 \Phi^2 h_3 : - 3\sqrt{\frac{3}{2}} : \Phi^1 \Phi^3 h_1 : \\
& - \frac{3}{2}\sqrt{\frac{3}{2}} : \Phi^1 \Phi^3 h_2 : + \frac{3}{2}\sqrt{\frac{3}{2}} : \Phi^1 \Phi^3 h_3 : - 3\sqrt{\frac{3}{2}} : \Phi^2 f_1 : + 3\sqrt{6} : \Phi^2 \Phi^3 h_1 : \\
& + 3\sqrt{\frac{3}{2}} : \Phi^2 \Phi^3 h_2 : + 3\sqrt{\frac{3}{2}} : \Phi^2 \Phi^3 h_3 : - 2\sqrt{6} : \Phi^2 \partial \Phi^3 : + 3\sqrt{\frac{3}{2}} : \Phi^3 f_1 : \\
& - \sqrt{6} : \partial \Phi^1 \Phi^2 : + \sqrt{6} : \partial \Phi^1 \Phi^3 : - 2\sqrt{6} : \partial \Phi^2 \Phi^3 : + \sqrt{\frac{3}{2}} \partial h_2 - \sqrt{\frac{3}{2}} \partial h_3,
\end{aligned}$$

$$\begin{aligned}
X = & -3f_{23} - \frac{3}{8} : h_2 h_2 : - \frac{3}{4} : h_2 h_3 : - \frac{3}{8} : h_3 h_3 : - \frac{3}{2} : \Phi^1 \Phi^2 h_2 : \\
& - \frac{3}{2} : \Phi^1 \Phi^2 h_3 : - \frac{3}{2} : \Phi^1 \Phi^3 h_2 : - \frac{3}{2} : \Phi^1 \Phi^3 h_3 : - \frac{1}{2} : \Phi^1 \partial \Phi^2 : \\
& - \frac{1}{2} : \Phi^1 \partial \Phi^3 : - 6 : \Phi^2 f_3 : + 3 : \Phi^2 \Phi^3 h_2 : - 3 : \Phi^2 \Phi^3 h_3 : + 5 : \Phi^2 \partial \Phi^3 : \\
& - 6 : \Phi^3 f_2 : + \frac{5}{2} : \partial \Phi^1 \Phi^2 : + \frac{5}{2} : \partial \Phi^1 \Phi^3 : - 5 : \partial \Phi^2 \Phi^3 : + \frac{1}{2} \partial h_2 + \frac{1}{2} \partial h_3,
\end{aligned}$$

$$\begin{aligned}
M = & -3\sqrt{\frac{3}{2}}f_{123} + 3\sqrt{\frac{3}{2}} : h_1 f_2 : + 3\sqrt{\frac{3}{2}} : h_1 f_3 : - \frac{3}{2}\sqrt{\frac{3}{2}} : h_2 f_1 : + 3\sqrt{\frac{3}{2}} : h_2 f_3 : \\
& - \frac{3}{2}\sqrt{\frac{3}{2}} : h_3 f_1 : + 3\sqrt{\frac{3}{2}} : h_3 f_2 : + 3\sqrt{6} : \Phi^1 f_{23} : - \frac{3}{2}\sqrt{\frac{3}{2}} : \Phi^1 h_1 h_2 : \\
& - \frac{3}{2}\sqrt{\frac{3}{2}} : \Phi^1 h_1 h_3 : - 3\sqrt{\frac{3}{2}} : \Phi^1 \Phi^2 f_1 : + 3\sqrt{\frac{3}{2}} : \Phi^1 \Phi^2 f_2 : + 9\sqrt{\frac{3}{2}} : \Phi^1 \Phi^2 f_3 : \\
& - \sqrt{6} : \Phi^1 \Phi^2 \partial \Phi^3 : - 3\sqrt{\frac{3}{2}} : \Phi^1 \Phi^3 f_1 : + 9\sqrt{\frac{3}{2}} : \Phi^1 \Phi^3 f_2 : + 3\sqrt{\frac{3}{2}} : \Phi^1 \Phi^3 f_3 : \\
& + \sqrt{6} : \Phi^1 \partial \Phi^2 \Phi^2 : + \sqrt{6} : \Phi^1 \partial \Phi^2 \Phi^3 : + \sqrt{6} : \Phi^1 \partial \Phi^3 \Phi^3 : - \frac{1}{2}\sqrt{\frac{3}{2}} : \Phi^1 \partial h_1 : \\
& + 3\sqrt{6} : \Phi^2 f_{13} : + 3\sqrt{\frac{3}{2}} : \Phi^2 h_1 h_2 : + 3\sqrt{\frac{3}{2}} : \Phi^2 h_2 h_3 : + 3\sqrt{6} : \Phi^2 \Phi^3 f_2 : \\
& - 3\sqrt{6} : \Phi^2 \Phi^3 f_3 : - 2\sqrt{6} : \Phi^2 \partial \Phi^3 \Phi^3 : - \frac{5}{2}\sqrt{\frac{3}{2}} : \Phi^2 \partial h_2 : + 3\sqrt{6} : \Phi^3 f_{12} : \\
& + 3\sqrt{\frac{3}{2}} : \Phi^3 h_1 h_3 : + 3\sqrt{\frac{3}{2}} : \Phi^3 h_2 h_3 : - \frac{5}{2}\sqrt{\frac{3}{2}} : \Phi^3 \partial h_3 : + \frac{5}{2}\sqrt{\frac{3}{2}} : \partial \Phi^1 h_1 : \\
& + \sqrt{\frac{3}{2}} : \partial \Phi^1 h_2 : + \sqrt{\frac{3}{2}} : \partial \Phi^1 h_3 : + \sqrt{6} : \partial \Phi^1 \Phi^1 \Phi^2 : + \sqrt{6} : \partial \Phi^1 \Phi^1 \Phi^3 : \\
& - \sqrt{6} : \partial \Phi^2 h_1 : + \frac{1}{2}\sqrt{\frac{3}{2}} : \partial \Phi^2 h_2 : - \sqrt{6} : \partial \Phi^2 h_3 : - 2\sqrt{6} : \partial \Phi^2 \Phi^2 \Phi^3 : \\
& - \sqrt{6} : \partial \Phi^3 h_1 : - \sqrt{6} : \partial \Phi^3 h_2 : + \frac{1}{2}\sqrt{\frac{3}{2}} : \partial \Phi^3 h_3 : - \frac{1}{2}\sqrt{\frac{3}{2}} \partial f_1 \\
& - \frac{5}{2}\sqrt{\frac{3}{2}} \partial f_2 - \frac{5}{2}\sqrt{\frac{3}{2}} \partial f_3 - \sqrt{\frac{2}{3}} \partial^2 \Phi^1 + \sqrt{\frac{2}{3}} \partial^2 \Phi^2 + \sqrt{\frac{2}{3}} \partial^2 \Phi^3.
\end{aligned}$$

The free field realization of $W_k(\mathfrak{g}, x, f)$ inside $V_k(\mathfrak{h}) \otimes F(\mathfrak{g}_{1/2})$ is induced by the canonical homomorphism $\mathfrak{g}_{\leq} \rightarrow \mathfrak{g}_0$, then we simply obtain the free field realization by removing the terms that contains a current $v \in \mathfrak{g}_{\leq} \setminus \mathfrak{g}_0$, i.e., the terms containing f 's.

For example in the case $\{\alpha = 1, k = -2/3\}$ the generators G and Φ look as:

$$G = \sqrt{\frac{3}{2}} : \Phi^1 h_1 : + \sqrt{\frac{3}{2}} : \Phi^2 h_2 : + \sqrt{\frac{3}{2}} : \Phi^3 h_3 : - \sqrt{\frac{2}{3}} \partial \Phi^1 - \sqrt{\frac{2}{3}} \partial \Phi^2 - \sqrt{\frac{2}{3}} \partial \Phi^3,$$

$$\Phi = -6 : \Phi^1 \Phi^2 \Phi^3 : + 3 : \Phi^2 h_2 : - 3 : \Phi^3 h_3 : - 2 \partial \Phi^2 + 2 \partial \Phi^3. \quad (3.4)$$

Therefore we have proved:

Theorem 3.1. Let $V_{-2/3}(\mathfrak{h})$ be the affine vertex algebra of level $-2/3$ associated to \mathfrak{h} with bilinear form A , and $F(\mathfrak{g}_{1/2})$ the vertex algebra of neutral free fermions as defined above. The vectors G and Φ given by the expressions above generate the $SW(\frac{3}{2}, \frac{3}{2}, 2)$ vertex algebra with $c = 21/2$ and $\varepsilon = 0$ inside $V_{-2/3}(\mathfrak{h}) \otimes F(\mathfrak{g}_{1/2})$. This vertex algebra is not simple and dividing by the ideal (B.2) we obtain the Shatashvili-Vafa G_2 superconformal algebra.

Remark 3.2. Note that $V_{-2/3}(\mathfrak{h}) \otimes F(\mathfrak{g}_{1/2})$ is isomorphic (by a linear transformation on the generators) to the vertex algebra of three free Bosons and three free Fermions with inner product minus the inverse of Cartan matrix (3.1) of $D(2, 1; 1) \simeq osp(4|2)$.

Remark 3.3. This free field realization was found by Mallwitz [10] using the most general ansatz on three free superfields of conformal weights $\frac{1}{2}$. By obtaining this realization from the quantum Hamiltonian formalism we can find explicitly the screening operators associated with the reduction as follows.

First we rescale the currents $h \in V_k(\mathfrak{h})$ and consider instead $\bar{h} := \frac{h}{\sqrt{k}}$, therefore $V_k(\mathfrak{h})$ is identified as a vertex algebra with the Heisenberg algebra $V_1(\mathfrak{h})$ associated to \mathfrak{h} .

Let V_Q denote the lattice vertex algebra [6] associated to the root lattice Q (that correspond to the Cartan matrix that we have fixed at the beginning of the section) of $D(2, 1; \alpha)$, i.e., we have three odd simple roots $\{\alpha_1, \alpha_2, \alpha_3\}$. Then for every lattice element α we have a $V_1(\mathfrak{h})$ -module M_α and a vertex operator Γ_α which is an intertwiner of type $\binom{M_0}{M_0 M_\alpha}$, hence its zero mode maps $V_1(\mathfrak{h}) = M_0 \rightarrow M_\alpha$.

Let $M_{-\alpha_i/\sqrt{k}}$ be the $V_1(\mathfrak{h})$ -module with highest weight $-\alpha_i/\sqrt{k}$ and $\Gamma_{-\alpha_i/\sqrt{k}}$ the intertwiner constructed just as in the lattice case, so that

$$\left[\bar{h}_i \Gamma_{-\alpha_j/\sqrt{k}} \right] = -\frac{(\alpha_i, \alpha_j)}{\sqrt{k}} \Gamma_{-\alpha_j/\sqrt{k}}, \quad \partial \left(\Gamma_{-\alpha_j/\sqrt{k}} \right) = -\bar{h}_j \Gamma_{-\alpha_j/\sqrt{k}}.$$

Define the operators

$$Q_i =: \Phi_i \Gamma_{-\alpha_i/\sqrt{k}} : \in V_1(\mathfrak{h}) \otimes F(\mathfrak{g}_{1/2}) \rightarrow M_{-\alpha_i/\sqrt{k}} \otimes F(\mathfrak{g}_{1/2}), \quad i = 1, 2, 3.$$

A straightforward computation using [18] shows that

$$W_k(\mathfrak{g}, x, f) \simeq \bigcap_{i=1}^3 \text{Ker } Q_{i(0)} \subset V_1(h) \otimes F(\mathfrak{g}_{1/2}),$$

equals the free field realization of $W_k(\mathfrak{g}, x, f)$ inside $V_k(\mathfrak{h}) \otimes F(\mathfrak{g}_{1/2})$ that we have produced above.

Remark 3.4. In fact a similar result can be obtained for the quantum Hamiltonian reduction of any simple Lie superalgebra when the nilpotent f is *superprincipal*, that is, there exists an odd nilpotent $F \in \mathfrak{g}_{-1/2}$ with $[F, F] = f$ ($f \in \mathfrak{g}_{-1}$ being a principal nilpotent) and these two vectors together with x form part of a copy of $osp(1|2) \subset \mathfrak{g}$. Not all Lie superalgebras admit a superprincipal embedding, in particular, it is necessary to admit a root system with all odd simple roots. In this case, one takes $F = \sum_i e_{-\alpha_i}$ the sum of all simple root vectors. The list of simple Lie superalgebras admitting an $osp(1|2)$ superprincipal embedding consists of

$$\begin{aligned} sl(n \pm 1|n), \quad osp(2n \pm 1|2n), \quad osp(2n|2n), \\ osp(2n + 2|2n), \quad D(2, 1; \alpha) \end{aligned}$$

In these case we see that $\mathfrak{g}_{1/2}$ is naturally isomorphic to $\Pi\mathfrak{h}^*$ and we can form the Boson-Fermion system and the screening charges as above. The intersection of their kernels coincides with the quantum Hamiltonian reduction for generic levels.

A λ -brackets of the Shatashvili-Vafa G_2 superconformal algebra

$$[\Phi_\lambda \Phi] = \left(-\frac{7}{2}\right)\lambda^2 + 6X, \quad [\Phi_\lambda X] = -\frac{15}{2}\Phi\lambda - \frac{5}{2}\partial\Phi,$$

$$[X_\lambda X] = \frac{35}{24}\lambda^3 - 10X\lambda - 5\partial X, \quad [G_\lambda \Phi] = K,$$

$$[G_\lambda X] = -\frac{1}{2}G\lambda + M, \quad [G_\lambda K] = 3\Phi\lambda + \partial\Phi,$$

$$[G_\lambda M] = -\frac{7}{12}\lambda^3 + (L + 4X)\lambda + \partial X, \quad [\Phi_\lambda K] = -3G\lambda - 3\left(M + \frac{1}{2}\partial G\right),$$

$$[\Phi_\lambda M] = \frac{9}{2}K\lambda - \left(3 : G\Phi : -\frac{5}{2}\partial K\right), \quad [X_\lambda K] = -3K\lambda + 3(: G\Phi : -\partial K),$$

$$[X_\lambda M] = -\frac{9}{4}G\lambda^2 - \left(5M + \frac{9}{4}\partial G\right)\lambda + \left(4 : GX : -\frac{7}{2}\partial M - \frac{3}{4}\partial^2 G\right),$$

$$[K_\lambda K] = -\frac{21}{6}\lambda^3 + 6(X-L)\lambda + 3\partial(X-L),$$

$$[K_\lambda M] = -\frac{15}{2}\Phi\lambda^2 - \frac{11}{2}\partial\Phi\lambda + 3(:GK: + 2:L\Phi:),$$

$$\begin{aligned} [M_\lambda M] = & -\frac{35}{24}\lambda^4 + \frac{1}{2}(20X-9L)\lambda^2 + \left(10\partial X - \frac{9}{2}\partial L\right)\lambda + \left(\frac{3}{2}\partial^2 X \right. \\ & \left. - \frac{3}{2}\partial^2 L - 4:GM: + 8: LX: \right), \end{aligned}$$

$$[L_\lambda X] = -\frac{7}{24}\lambda^3 + 2X\lambda + \partial X, \quad [L_\lambda M] = -\frac{1}{4}G\lambda^2 + \frac{5}{2}M\lambda + \partial M.$$

B The $SW(\frac{3}{2}, \frac{3}{2}, 2)$ superconformal algebra

Here we follow the presentation in [14]. The $SW(\frac{3}{2}, \frac{3}{2}, 2)$ superconformal algebra has six generators $\{G, L, H, \tilde{M}, W, U\}$ where G and L generate the $N=1$ superconformal algebra of central charge c , and (H, \tilde{M}) and (W, U) are two superconformal multiplets of dimensions $\frac{3}{2}$ and 2 respectively.

A superconformal multiplet $\hat{\Phi} = (\Phi, \Psi)$ of dimension Δ is a pair of two primary fields of conformal weights Δ and $\Delta + \frac{1}{2}$ respectively, such that the λ -brackets with the supersymmetry generator G are as follow:

$$[G_\lambda \Phi] = \Psi, \quad [G_\lambda \Psi] = (\partial + 2\Delta\lambda)\Phi.$$

The other λ -brackets between the generators are as follow:

$$[H_\lambda H] = \frac{c}{3}\lambda^2 + \varepsilon\tilde{M} + 2L + \frac{4}{3}\mu W, \tag{B.1}$$

$$[H_\lambda \tilde{M}] = (3G + 3\varepsilon H)\lambda + \frac{-2}{3}\mu U + \partial G + \varepsilon\partial H,$$

$$[\tilde{M}_\lambda \tilde{M}] = \frac{1}{3}c\lambda^3 + (4\varepsilon\tilde{M} + 8L + \frac{4}{3}\mu W)\lambda + 2\varepsilon\partial\tilde{M} + 4\partial L + \frac{2}{3}\mu\partial W,$$

$$[H_\lambda W] = \mu H\lambda + \frac{\varepsilon}{2}U + \frac{\mu}{3}\partial H,$$

$$[\tilde{M}_\lambda W] = (\frac{\mu}{3}\tilde{M} + 2\varepsilon W)\lambda + \frac{9\mu}{2c}:GH: + \frac{\mu(-27+2c)}{12c}\partial\tilde{M} + \varepsilon\partial W,$$

$$\begin{aligned}
[H_\lambda U] &= \left(\frac{-2}{3}\mu\tilde{M} + 2\varepsilon W\right)\lambda + \frac{9\mu}{2c} : GH : - \frac{\mu(27+2c)}{12c}\partial\tilde{M} + \frac{\varepsilon}{2}\partial W, \\
[\tilde{M}_\lambda U] &= \mu H\lambda^2 + \left(\frac{5}{2}\varepsilon U + \frac{2}{3}\mu\partial H\right)\lambda - \frac{9\mu}{2c} : G\tilde{M} : + \frac{9\mu}{c} : LH : + \varepsilon\partial U + \frac{\mu(-27+2c)}{12c}\partial^2 H, \\
[W_\lambda W] &= \frac{c}{12}\lambda^3 + \left(2L + \frac{\varepsilon}{2}\tilde{M} + \frac{\mu(10c-27)}{6c}W\right)\lambda + \partial L + \frac{\varepsilon}{4}\partial\tilde{M} + \frac{\mu(10c-27)}{12c}\partial W,
\end{aligned}$$

$$\begin{aligned}
[W_\lambda U] &= \left(-\frac{3}{2}G - \frac{3}{4}\varepsilon H\right)\lambda^2 + \left(\frac{\mu(-27+10c)}{12c}U - \partial G - \frac{\varepsilon}{2}\partial H\right)\lambda \\
&\quad - \frac{1}{48c} \left(162\varepsilon : G\tilde{M} : + 432\mu : GW : - 324 : H\tilde{M} : + 648 : LG : \right. \\
&\quad \left. + 324\varepsilon : LH : - 8\mu(27+2c)\partial U + 6(-27+2c)\partial^2 G \right. \\
&\quad \left. + 3(-27+2c)\varepsilon\partial^2 H\right),
\end{aligned}$$

$$\begin{aligned}
[U_\lambda U] &= -\frac{c}{12}\lambda^4 - \left(\frac{5}{4}\varepsilon\tilde{M} + 5L + \frac{\mu(-27+10c)}{6c}W\right)\lambda^2 - \left(\frac{5}{4}\varepsilon\partial\tilde{M} + 5\partial L \right. \\
&\quad \left. + \frac{\mu(-27+10c)}{6c}\partial W\right)\lambda - \frac{1}{16c} \left(-144\mu : GU : - 108 : G\partial G : \right. \\
&\quad \left. - 54\varepsilon : G\partial H : + 108 : H\partial H : - 108 : \tilde{M}\tilde{M} : + 216\varepsilon : L\tilde{M} : \right. \\
&\quad \left. + 432 : LL : + 288\mu : LW : + 54\varepsilon : \partial GH : - 3(9-2c)\varepsilon\partial^2\tilde{M} \right. \\
&\quad \left. + 24c\partial^2 L - 4\mu(27-2c)\partial^2 W\right),
\end{aligned}$$

where $c, \varepsilon \in \mathbb{C}$ and $\mu = \sqrt{\frac{9c(4+\varepsilon^2)}{2(27-2c)}}$.

Remark B.1. For $(c, \varepsilon) = (\frac{21}{2}, 0)$ it was checked in [14] that $SW(\frac{3}{2}, \frac{3}{2}, 2)$ coincides with the Shatashvili-Vafa G_2 algebra at central charge $\frac{21}{2}$ modulo the ideal generated by:

$$2\sqrt{14} : GW : -3 : H\tilde{M} : +2 : LG : -2\sqrt{14}\partial U. \quad (\text{B.2})$$

The existence of this ideal was first observed in [5]. The relations between the generators of $SW(\frac{3}{2}, \frac{3}{2}, 2)$ in the case $(\frac{21}{2}, 0)$ and the generators of the Shatashvili-Vafa G_2 algebra as presented in the Appendix A are given by:

$$\Phi = iH, \quad K = i\tilde{M}, \quad X = -(L + \sqrt{14}W)/3, \quad M = -(\partial G + 2\sqrt{14}U)/6. \quad (\text{B.3})$$

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